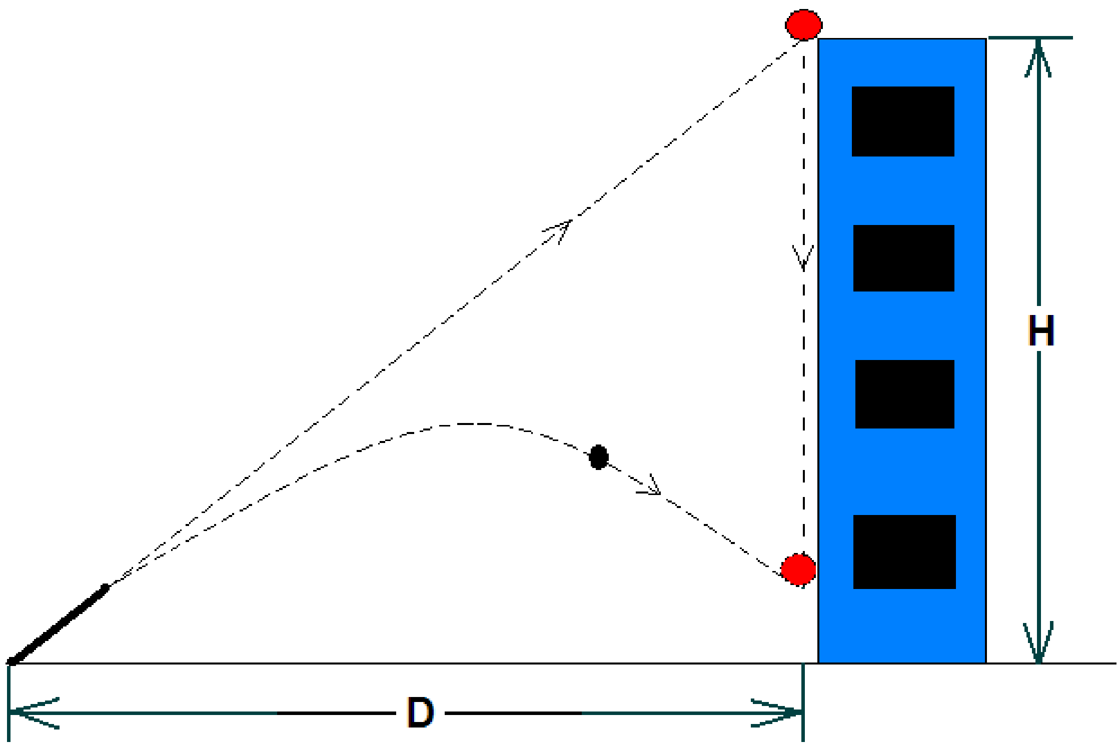
Problem :

A ball is dropped from a building of height (H). *At the same moment*, a projectile is fired from a horizontal distance, (D) from the building.

Assuming that you have control of the angle of elevation and the initial speed of the fired projectile, find a general expression for the height above the ground at which they collide.



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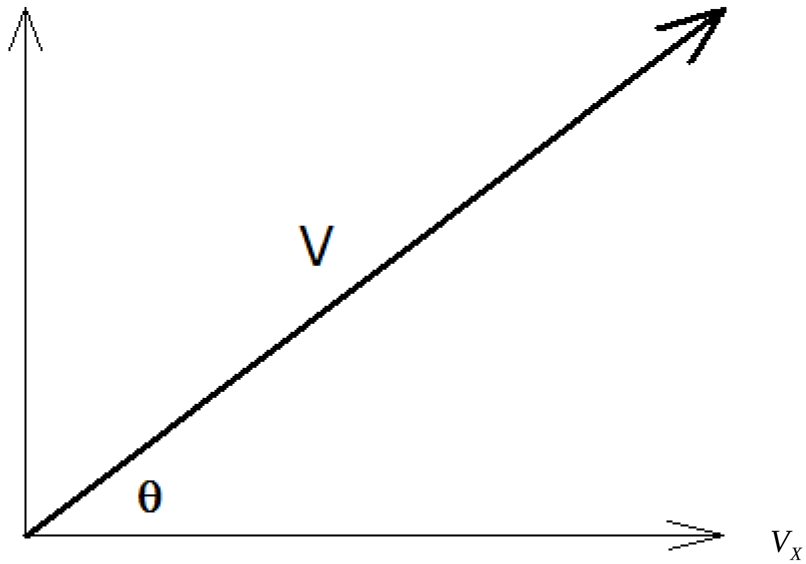
Solution :

The initial velocity of the projectile has vertical and horizontal components which are independent of one another:

V_Y



The horizontal velocity : $\frac{V_X}{V} = \cos(\theta) \quad \rightarrow \quad V_X = V \bullet \cos(\theta)$

The vertical velocity : $\frac{V_Y}{V} = \sin(\theta) \quad \rightarrow \quad V_Y = V \bullet \sin(\theta)$

The projectile and the dropped ball must arrive at the same place, at the same time.
Call the collision time (t_C), and call the height above ground at the collision $(z_{collision})$.

The horizontal distance (D) is related to (t_C) according to, $D = V_X \bullet t_C$
One expression for the collision time is, therefore;

$$t_C = \frac{D}{V \bullet \cos(\theta)}$$

While (V_Y) is initially a constant, the vertical component of velocity decreases until maximum
height is reached, and then it increases on the downward portion of the path.

The vertical height (z) at any time (t) is given *in general*, by…

$$z_{proj} = z_0 + V_Y \bullet t + \frac{1}{2}gt^2$$

For *our* projectile starting at ground level, take upward motion as (+) and downward

motion as (-). The second term on the right side is then (+) but the acceleration due
to gravity tries to impede that motion and is therefore (-). Since the initial position of
the projectile is the ground, then we have $z_0 = 0$.

The vertical height of the projectile at any later time (t) is given by…

$$z_{proj} = [V \bullet \sin(\theta)] \bullet t - \frac{1}{2} g t^2$$

Let's look at the position of the dropped ball as a function of time (t) after it was dropped.
The red ball's *horizontal* position never changes. Its vertical position in general, is…

$$z_{ball} = z_0 + U_Y \bullet t + \frac{1}{2} g t^2$$

But now, the initial position of the ball is, $z_0 = H$, and it's initial speed is $U_Y = 0$.
Since the ball's vertical position is decreasing as it moves downward, we have
the following for the vertical position of the ball at any later time, (t)…

$$z_{ball} = H - \frac{1}{2} g t^2$$

At the time of the collision (t_c) the vertical positions of the ball and projectile are the
same:

$$z_{ball} = z_{proj}$$

$$H - \frac{1}{2} g t_c^2 = [V \bullet \sin(\theta)] \bullet t_c - \frac{1}{2} g t_c{}^2$$

$$H = [V \bullet \sin(\theta)] \bullet t_c \qquad \rightarrow \qquad t_c = \frac{H}{V \bullet \sin(\theta)}$$

We now have two expressions for (t_C). Equating them, we get…

$$\frac{D}{V \bullet \cos(\theta)} = \frac{H}{V \bullet \sin(\theta)} \quad \rightarrow \quad \frac{\sin(\theta)}{\cos(\theta)} = \frac{H}{D} \quad \rightarrow$$

$$\tan(\theta) = \frac{H}{D}$$

$$\theta = \tan^{-1}\left(\frac{H}{D}\right)$$

This latter equation shows that the angle of elevation of the projectile depends only on
the fixed parameters, (H) and (D). Use this last equation and the sketch at the top of
the page, to show that if a collision is to occur, then initially, the projectile must be aimed
at the ball at the top of the building.

Now let's concentrate on the vertical location where the collision occurs.

The height of the projectile at collision occurs for (z_{ball}) above, when we replace (t) with (t_C) :

$$z_{ball} = H - \frac{1}{2} g t_c^{\ 2}$$

Substitute one of our two expressions for (t_C) :

$$z_{collision} = H - \frac{1}{2} g \left(\frac{H}{V \bullet \sin(\theta)} \right)^2$$

Using the first sketch, see that $\sin(\theta) = \frac{H}{\sqrt{H^2 + D^2}}$. Make this substitution and simplify…

$$z_{collision} = H - \frac{g \bullet (H^2 + D^2)}{2 \bullet V^2}$$

… and from earlier, $\theta = \tan^{-1}\left(\frac{H}{D}\right)$.

From these two equations one can tell a lot about the experiment.

Note for example, that the height at which the collision occurs, can be negative.
That means, the dropped ball hits the ground before the projectile reaches it.
The negative value indicates that the “collision” would have to happen underground.

Also note that for arbitrarily chosen values of (H) and (D), one cannot be equally
arbitrary in selecting the speed of the projectile.

Try some numbers !