Problem :

An observer standing at a street corner looks up and sees that the stoplight is red, for the main street traffic. A (special) car on the main street is approaching the intersection at a constant speed, 20% of the speed of light.

Show that the car's driver sees the same stoplight, as green !

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Solution :

This car is receiving light waves, while moving at a relativistically high speed. The car will intercept more waves per second, than it would if it were stopped. Therefore the frequency of light seen by the driver should increase (and correspondingly, the wavelength should decrease).

We can use the relativistic Doppler formula for light waves to determine the wavelength of light (hence its color) determined by the driver in the moving car.

$$\lambda_{car} = \lambda_o \bullet \sqrt{\frac{1+(V/c)}{1-(V/c)}} \quad ; \quad \text{where,}$$

λ_{car} = wavelength of light seen by the *moving* car

λ_o = wavelength of light seen by the *observer standing* on the corner

V = speed of the oncoming car, $(0.20 \bullet c)$

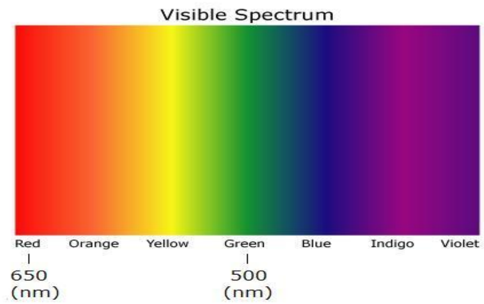
c = speed of light.

As written, this formula applies to a car moving *away* from the observer (i.e., along the +x axis). For our situation, the car is *approaching* the observer, so change the sign on v $(+ \rightarrow -)$

$$\lambda_{car} = 650 \bullet \sqrt{\frac{1+(-0.20)}{1-(-0.20)}}$$; using the red light wavelength as 650 nm.

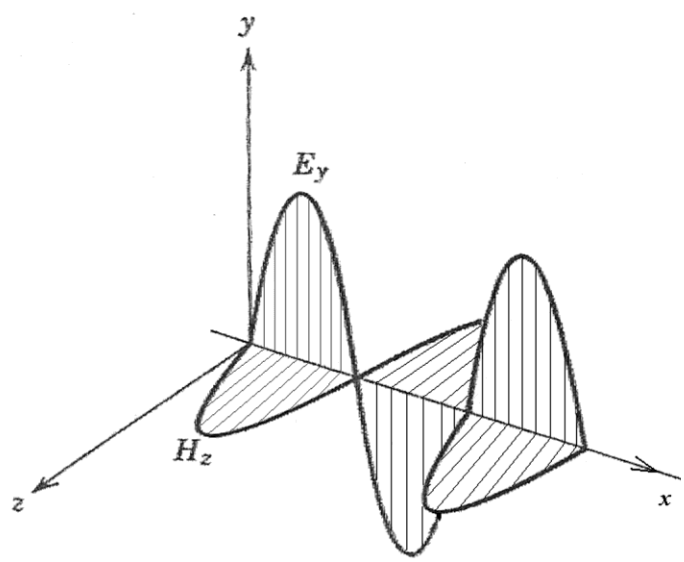
$\lambda_{car} =$ 531 nm.

Since a 531 nm wavelength is characteristic of green light, the stoplight appears green to the driver, while appearing red to the person standing on the corner…



There are several ways to get the formula we used to solve this problem. Here is one of the more interesting ways of obtaining it :

We'lll work with the wave nature of light. For light traveling along the +X-axis, the electric and magnetic fields comprising it, looks like this…



For the observer standing at the stoplight, the electromagnetic wave has these components:

$$E_X = 0 \qquad H_X = 0$$

$$E_Y = A \bullet \sin\left[\omega\left(t - \frac{x}{c}\right)\right] \qquad H_Y = 0$$

$$E_Z = 0 \qquad H_Z = A \bullet \sin\left[\omega\left(t - \frac{x}{c}\right)\right]$$

The electric field component (E_Y) and the magnetic field component (H_Z) have the typical form for a traveling wave moving along the positive X-axis with frequency (ω) and speed (c). Note from the arguments of the sine functions, that the two waves are in-phase. Also notice that there are no field components *in the direction* of motion of the wave (i.e., these are transverse waves).

The Lorentz transformations give the electric and magnetic field components in the coordinate system moving at speed (V)…We'll need only focus on one of the fields to obtain the Doppler shift in frequency due to the motion of the observer. The moving observer then, sees the following electric field components, designated with the 'primes':

$$E'_X = E_X \qquad E'_Y = \frac{E_Y - \left(V/c\right) \bullet H_Z}{\sqrt{1 - \left(V/c\right)^2}} \qquad E'_Z = \frac{E_Z + \left(V/c\right) \bullet H_Y}{\sqrt{1 - \left(V/c\right)^2}}$$

Now substitute into these expressions, the unprimed coordinate values above…

$$E'_X = 0$$

$$E'_Y = \frac{A \bullet \sin\left[\omega\left(t - \frac{x}{c}\right)\right] - \left(V/c\right) \bullet A \bullet \sin\left[\omega\left(t - \frac{x}{c}\right)\right]}{\sqrt{1 - \left(V/c\right)^2}}$$

$$E'_Z = \frac{0 + \left(V/c\right) \bullet (0)}{\sqrt{1 - \left(V/c\right)^2}} = 0$$

The x- and z- components are zero, and the moving observer sees the modified y-component. Factor out the similar terms in the numerator to show that…

$$E'_Y = A \bullet \sin\left[\omega\left(t - \frac{x}{c}\right)\right] \bullet \left\{\frac{1 - \left(V/c\right)}{\sqrt{1 - \left(V/c\right)^2}}\right\}$$; *you* can simplify this, to get…

$$E'_Y = A \bullet \sqrt{\frac{1-\left(V/c\right)}{1+\left(V/c\right)}} \bullet \sin\left[\omega\left(t-\frac{x}{c}\right)\right]$$

The moving observer sees a change in amplitude (by that square root factor) in the electric field. But this formula does not show any change in the frequency of the light due to the motion of the observer. That's because the (primed) moving observer's equation for the field, has the stationary observers parameters for time (t) and position (x). We need to insert primed values for the time and position, in order to have a self-consistent expression for the primed field. We need…

$$t-\left(\frac{x}{c}\right) \quad \rightarrow \quad t'-\left(\frac{x'}{c}\right)$$

The 'normal' Lorentz transformations give the transition from primed-to-unprimed coordinate systems. Here we have to use the inverse Lorentz transformations to go from unprimed-to – primed observers (remember, the unprimed observer is standing at the stoplight, and the primed observer is moving in the car).

It's also important to note in the above transformation, that the speed of light is the same in both the primed system and the unprimed system. That's a fundamental postulate of the Special Theory of relativity.

The inverse Lorentz transformations for (x) and for (t) are given by…

$$x = \frac{x' + V \bullet t'}{\sqrt{1-\left(V/c\right)^2}} \qquad \text{and} \qquad t = \frac{t' + \left(V/c^2\right) \bullet x'}{\sqrt{1-\left(V/c\right)^2}} \qquad \text{then…}$$

$$t-\left(\frac{x}{c}\right) = \frac{1}{\sqrt{1-\left(V/c\right)^2}} \bullet \left\{t' + \left(V/c^2\right) \bullet x' - \left(\frac{x'}{c}\right) - \left(V/c\right) \bullet t'\right\} \quad \text{which simplifies to…}$$

$$t-\left(\frac{x}{c}\right) = \frac{1}{\sqrt{1-\left(V/c\right)^2}} \bullet \left\{\left[1-\left(V/c\right)\right] \bullet t' - \left(\frac{x'}{c}\right) \bullet \left[1-\left(V/c\right)\right]\right\}.$$

Factor the numerator and simplify to show that…

$$t-\left(\frac{x}{c}\right) = \sqrt{\frac{1-\left(V/c\right)}{1+\left(V/c\right)}} \bullet \left[t' - \left(\frac{x'}{c}\right)\right].$$ Now substitute this into the last equation for (E'_Y)..

Finally then,

$$E'_Y = A \bullet \sqrt{\frac{1-\left(V/c\right)}{1+\left(V/c\right)}} \bullet \sin\left[\omega \bullet \sqrt{\frac{1-\left(V/c\right)}{1+\left(V/c\right)}} \bullet \left(t' - \frac{x'}{c}\right)\right]$$

From this expression, see that the motion alters the amplitude and the frequency by the same factor (i.e., the square root term). For our Doppler problem, we're just concerned for the frequency part of the above equation :

The 'primed' frequency is,

$$\omega' = \omega \bullet \sqrt{\frac{1-\left(V/c\right)}{1+\left(V/c\right)}}$$. Convert these radians/sec to cycles/sec , using $\omega = 2\pi\nu$.

$$\nu' = \nu \bullet \sqrt{\frac{1-\left(V/c\right)}{1+\left(V/c\right)}}$$ … and convert to wavelength, using $\nu\lambda = c = \nu'\lambda'$.

Noting the *inverse relation* between frequency (ν) and wavelength (λ), simplify to finally get,

$$\lambda' = \lambda \bullet \sqrt{\frac{1+\left(V/c\right)}{1-\left(V/c\right)}}$$; where $\lambda' = \lambda_{car}$ and $\lambda = \lambda_o$ in the very first equation of this problem.