Problem:

Find the cube root(s) of the number, 1.

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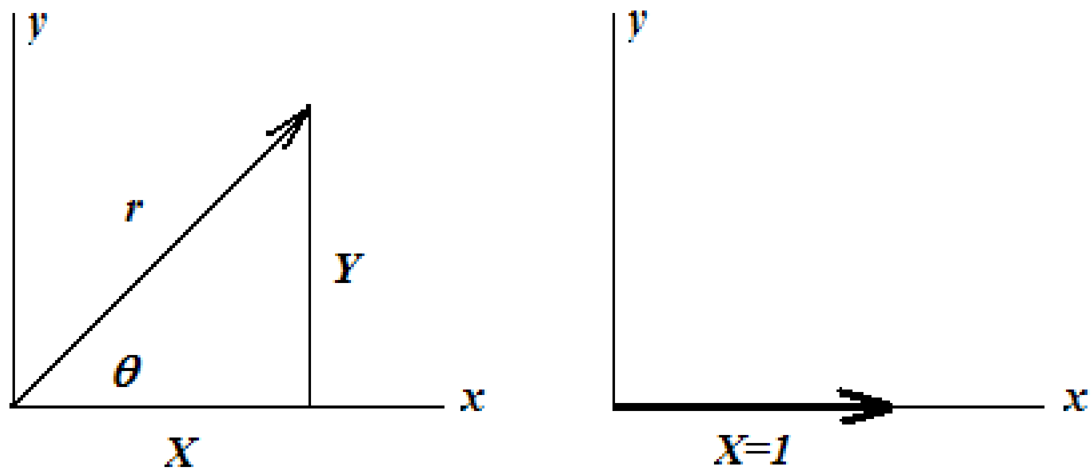
Solution:

Expressed on an Argand diagram, the real number (1) is represented as a *horizontal* line from the origin to the point, x=1.



The number (1) the *real* part of the complex number, (X + iY),
Where X =1 is the real part and here, the *imaginary* part has Y=0.

From the diagram, $X = r \bullet Cos(\theta)$ and $Y = r \bullet Sin(\theta)$
So,

$$X + iY = r[Cos(\theta) + i \bullet Sin(\theta)]$$

The right side of the latter equation is usually abbreviated, $Cis(\theta)$.

Since (r) is the *horizontal* unit vector of *length*, 1, we can represent the number (1) as…

$1 + i(0) = Cos(0) + i \bullet Sin(0)$; but for now let's keep the angle as θ …

Take the cube root of both sides…

$\sqrt[3]{1} = [Cis(\theta)]^{(1/3)}$: now apply De Moivre's Theorem

$$\sqrt[3]{1} = \left[Cis\left(\frac{\theta}{3}\right) \right]$$

But on the unit circle, one can traverse complete circles, increasing the angle by (2π) radians each time, ***but arriving back at the original angle each time***. If we make (n) complete traversals of (2π) each, then our angle is more generally written as… $\theta \to \theta + n(2\pi)$. Our cube root becomes,

$$\sqrt[3]{1} = \left[Cis\left(\frac{\theta + n(2\pi)}{3}\right) \right]$$; now let's re-set our angle to $\theta = 0$.

$$\sqrt[3]{1} = \left[Cis\left(\frac{n(2\pi)}{3}\right) \right]$$; n=0,1,2,3,…

Let's take n=0 :

$$\sqrt[3]{1} = \left[Cis\left(\frac{0(2\pi)}{3}\right) \right]$$

$$\sqrt[3]{1} = Cos(0) + i \bullet Sin(0) \quad \to \quad 1 + i \bullet (0)$$

$\sqrt[3]{1} = 1$; OK, this is the first result- the one we expected.

Now let's try n=1…

$$\sqrt[3]{1} = \left[Cis\left(\frac{1(2\pi)}{3}\right) \right]$$

$$\sqrt[3]{1} = Cos\left(\frac{2\pi}{3}\right) + i \bullet Sin\left(\frac{2\pi}{3}\right)$$; use the conversion $\frac{2\pi}{3} \to 120^{\circ}$

$$Cos(120) = Cos(90 + 30) \to Cos(90)Cos(30) - Sin(90)Sin(30)$$

$$Cos(120) = 0 - (1) \bullet \left(\frac{1}{2} \right) \rightarrow \frac{-1}{2}$$

$$Sin(120) = Sin(90 + 30) \rightarrow Sin(90)Cos(30) + Cos(90)Sin(30)$$

$Sin(120) = (1) \bullet \frac{\sqrt{3}}{2} + 0 \rightarrow \frac{\sqrt{3}}{2}$; put all together, we get…

$\sqrt[3]{1} = -\frac{1}{2} + i \bullet \frac{\sqrt{3}}{2}$; this is the second result.

Now you try the case for n=2. You should get this third result…

$$\sqrt[3]{1} = -\frac{1}{2} - i \bullet \frac{\sqrt{3}}{2}$$

If you then try the case for n=3, show that you get the same result obtained for n=0. After the n=2 case, the higher (n) values repeatedly generate the three results obtained here- so there are only three cube roots of the number, 1… two imaginary and one real.