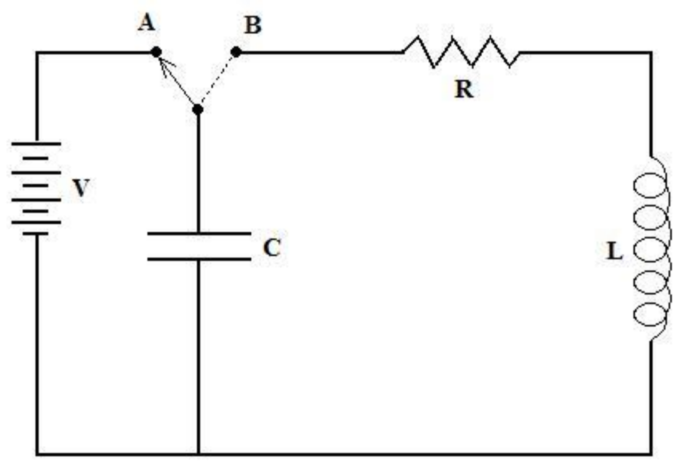
Problem :

A circuit has a resistance (R) and a capacitor (C) and an inductor (L) as shown. The capacitor can be charged to voltage (V) by throwing the switch to position (A).

With the capacitor charged, the switch is thrown to position (B). Show that values exist for (R) and (C) and (L) such that the current flowing in the closed circuit is an Alternating Current.



Solution :

Since the capacitance (C) is defined by, $C = \dfrac{Q}{V}$, where (Q) is the charge on the plates of the capacitor, then the voltage drop across the capacitor is, $V_C = \dfrac{Q}{C}$

The voltage drop across the resistor (R) is, $V_R = IR$, where (I) is the current flowing in the circuit when the switch is thrown to position-B.

The voltage drop across the inductor is given by Faraday's Law… $V_L = L\dfrac{dI}{dt}$; that is, the voltage drop is proportional to the time rate of change of current flowing in the loop.

The total voltage drop across this circuit is zero, since there is no supply in the loop…

$V_L + V_C + V_R = 0$; substitute the proper expressions…

$L\dfrac{dI}{dt} + RI + \dfrac{1}{C}q = 0$; but from the definition of current… $I = \dfrac{dq}{dt}$ we get…

$$L\frac{d}{dt}\left(\frac{dq}{dt}\right)+R\frac{dq}{dt}+\frac{1}{C}q=0$$; Simplify this …

$$\frac{d^2q}{dt^2}+\left(\frac{R}{L}\right)\frac{dq}{dt}+\left(\frac{1}{LC}\right)q=0$$.

This is a second order differential equation with constant coefficients.

Try a solution of the form, $q=a\varepsilon^{bt}$; Substitute this into the above equation, and simplify, including factoring common terms. Show that you get this result…

$$a\varepsilon^{bt}\left[b^2+\left(\frac{R}{L}\right)b+\left(\frac{1}{LC}\right)\right]=0$$

The quantity in the brackets must be zero, to avoid a trivial solution. Solve for (b)…

$$b=\frac{-\left(\frac{R}{L}\right)\pm\sqrt{\left(\frac{R}{L}\right)^2-\frac{4}{LC}}}{2}$$; Let's require that $\frac{4}{LC}>\left(\frac{R}{L}\right)^2$

This latter requirement gives a complex numerator, and has the potential for leading to an oscillatory solution. Therefore…

$$b=\frac{-\left(\frac{R}{L}\right)\pm i\bullet\sqrt{\frac{4}{LC}-\left(\frac{R}{L}\right)^2}}{2}\quad\rightarrow\quad b=-\left(\frac{R}{2L}\right)\pm i\bullet\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}$$

For simplicity, temporarily let $\beta=\left(\frac{R}{2L}\right)$ and $\omega=\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}$

Using both values for (b), our general solution becomes…

$$q(t)=a\varepsilon^{-\beta t+i\omega t}+a\varepsilon^{-\beta t-i\omega t}\quad\rightarrow\quad q(t)=a\varepsilon^{-\beta t}\left[\varepsilon^{i\omega t}+\varepsilon^{-i\omega t}\right]$$

Now use the definition, $Cos(\omega\bullet t)=\frac{\varepsilon^{i\omega t}+\varepsilon^{-i\omega t}}{2}$, then…

$$q(t)=2a\varepsilon^{-\beta t}Cos(\omega\bullet t)$$

To get (a), use the initial condition that at (t = 0), the charge on the capacitor is given by,

$$C = \frac{q(0)}{V} \quad \rightarrow \quad q(0) = CV \ . \quad \text{Setting} \quad (t{=}0),$$

$CV = 2a\varepsilon^{0}Cos(0) \quad \rightarrow \quad CV = 2a$. The solution for the charge in the circuit is,

$$q(t) = CV\varepsilon^{-\beta t}Cos(\omega \bullet t)$$

To get the current, note as we did earlier, that $I = \frac{dq}{dt}$; Take the derivative and simplify…

$$I(t) = -CV\varepsilon^{-\beta t}\left[(\beta)Cos(\omega \bullet t) + (\omega)Sin(\omega \bullet t)\right]$$

Remember the constraint for this oscillatory current,

$$\frac{4}{LC} > \left[\frac{R}{L}\right]^2 \quad \rightarrow \quad \frac{R^2C}{L} < 4$$

In the equation above for the current, the term in the brackets governs the oscillatory nature of the current; the term to the left of the brackets is a negative exponential- revealing that the current eventually dies out with increasing time.

Let's try a set of parameters, subject to our constraint… Try these ;

R = 0.1 ohms   L = 0.05 henrys   C= 0.0001 farads

Test the constraint :

$$\frac{R^2C}{L} < 4 \quad \rightarrow \quad \frac{(0.1)^2 \bullet (0.0001)}{(0.05)} < 4 \quad \rightarrow \quad 0.00002 < 4 \ : \ \textbf{OK}$$

Compute (β) and the frequency, (ω) :

$$\beta = \left(\frac{R}{2L}\right) \quad \rightarrow \quad \beta = \frac{0.10}{2 \bullet 0.05} \quad \rightarrow \quad \beta = 1$$

$$\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \quad \rightarrow \quad \omega = \sqrt{\frac{1}{0.05 \bullet 0.0001} - \left(\frac{0.1}{2 \bullet 0.05}\right)^2}$$

$$\omega = 447.2 \quad \text{rad/sec} \quad \rightarrow \quad \omega \approx 71 \quad \text{cyc/sec}$$

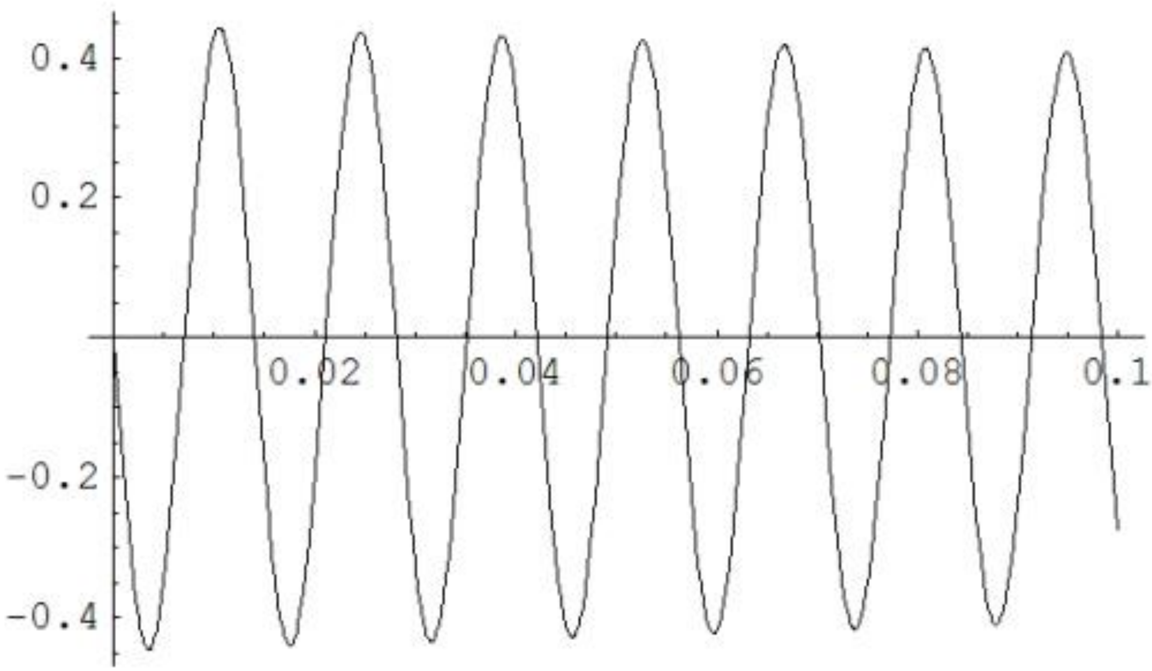
This frequency (not unlike household AC in the US) indicates a current oscillation of 71 times, each second. This corresponds to about 7.1 cycles per tenth of a second.

Let's plot the current (vertical axis) vs time (horizontal axis) to see if we get this 7 cycles in about 0.1 second, and to see if there is a decay in the intensity with time.

Using our computed values of (β) and (ω) the current is…

$$I(t) = -(0.001)\varepsilon^{-t}\left[Cos(447.2 \bullet t) + 447.2 \bullet Sin(447.2 \bullet t)\right]$$

When plotted, here's the result…



See that for a time of 0.1 second, there are indeed, 7 cycles.

Also note that even over this short time span, you can see a slight decay in the peak magnitude of the current,

What units should be applied to the current (vertical axis)?

What would you have to do to inhibit the decay of the current with time?

How could you change the frequency of the oscillations?