Problem :

A cup of hot coffee is placed on a table in a kitchen having well-insulated walls.

Why does the cup of coffee cool down, until it reaches room temperature?

Why doesn't heat leave the active air molecules of the room, and enter the coffee… causing the coffee to attain even a higher temperature?

Give a quantitative response.

Solution :

The reason the heat preferentially leaves the coffee (cooling it down) lies in the Second Law of Thermodynamics.

There are many ways to express this Law. Here are two expressions of it…

A) Heat always flows from where it is, to where it isn't (i.e., from a hotter place to a cooler place).

B) In any **isolated** system (e.g., the insulated kitchen) the ***entropy*** of the system always **increases**.

Let's assume that the heat leaves the coffee. We then must show that the entropy of the system (consisting of the whole kitchen) indeed, does ***increase***.

The heat yielded by the coffee is given by…

$\Delta Q = m \bullet c \bullet (T_H - T_{room})$, where (m) = mass of coffee in the cup

and, (c) = heat capacity of the liquid

and, (T_H) = initial temperature of the liquid

and, (T_{room}) = temperature in kitchen (assumed constant)

The entropy change in the room (kitchen) is given by…

$$\Delta S_{room} = \frac{\Delta Q}{T_{room}} \quad \rightarrow \quad \Delta S_{room} = \frac{m \bullet c \bullet (T_H - T_{room})}{T_{room}} \quad \rightarrow \quad m \bullet c \bullet \left(\frac{T_H}{T_{room}} - 1 \right)$$

The coffee slowly cools from (T_H) to (T_{room}). Its change in entropy is given by…

$$\Delta S_{coffee} = \int_{T_H}^{T_{room}} \frac{m \bullet c \bullet dT}{T} \quad \rightarrow \Delta S_{coffee} = m \bullet c \bullet \int_{T_H}^{T_{room}} \frac{dT}{T} \quad \rightarrow \quad m \bullet c \bullet Ln\left(\frac{T_{room}}{T_H} \right)$$

The total entropy change of the system, is given by…

$\Delta S_{total} = \Delta S_{room} + \Delta S_{coffee}$. *You* show that this total entropy can be written as…

$$\Delta S_{total} = m \bullet c \bullet \left[\left(\frac{T_H}{T_{room}} - 1 \right) - Ln\left(\frac{T_H}{T_{room}} \right) \right].$$

The whole problem now comes down to showing that this last expression is always positive. You could do it graphically, but an analytical method applicable to all temperatures, is preferable.

Since (m) and (c) are already positive, focus attention on the term in brackets.

The problem tells us that $T_H > T_{room}$. Therefore, $\dfrac{T_H}{T_{room}} > 1$.

Define a variable (x > 0) such that… $\varepsilon^x = \dfrac{T_H}{T_{room}}$. The total entropy change is then…

$$\Delta S_{total} = m \bullet c \bullet \left[\left(\varepsilon^x - 1 \right) - Ln\left(\varepsilon^x \right) \right] \quad \rightarrow \quad \Delta S_{total} = m \bullet c \bullet \left[\left(\varepsilon^x - 1 \right) - x \right]$$

The series expansion of the exponential is given by…

$$\varepsilon^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \rightarrow \quad \varepsilon^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

Now substitute this series, and simplify the term in brackets…

$\Delta S_{total} = m \bullet c \bullet \left[\dfrac{x^2}{2} + \dfrac{x^3}{6} + \ldots \right]$. *Every term is positive*, all the time. Therefore,

$$\Delta S_{total} > 0.$$

So, in order for the heat to have left the coffee cup, we see the entropy had to increase. The direction of the flow of heat is consistent with the Second Law of Thermodynamics.

Note the validity of the logically equivalent contrapositive statement. Namely, "…if the entropy does not increase, then heat does not flow from the coffee to the room."