Problem :

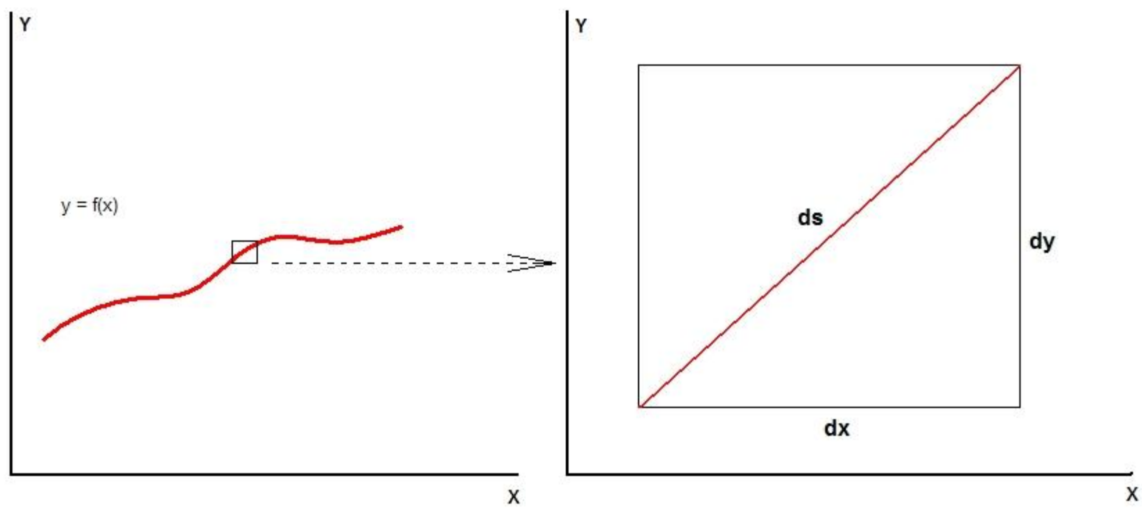
Starting from the definition of a circle, prove that the circumference of a circle is $C = 2\pi R$, that is, show that the ratio of the circumference to the diameter of any circle is the constant, π (in flat space).

Solution :

When we proved the area formula for a circle, we showed that the equation of a circle centered at the origin of the coordinate system, is given by,

$$x^2 + y^2 = R^2 \qquad \text{or,} \quad y = \pm\sqrt{R^2 - x^2}$$

First, let's find a general formula for the path length along the curve of any function, y = f(x) .



The small box (having dimensions… Δx and Δy) encompasses a portion of the curve.

If the dimensions of the box were made to be infinitely small, then …

$\Delta x \to dx$ and $\Delta y \to dy$; as shown in the vastly magnified view on the right.

In this limit of an infinitesimally small square, the tiny path length has length, (ds).

By Pythagoras…

$$(ds)^2 = (dx)^2 + (dy)^2 \qquad \to \qquad (ds)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right] \bullet (dx)^2$$

Notice that the quantity in brackets is not the second derivative…it is the *square of the first derivative*…

Take the square root of both sides of this last equation, and sum over all the infinitesimal's, (dx);

$$\int ds = s \quad \rightarrow \quad \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

For our situation, integrate only over the domain, x =0 to x=R; that will give us the circumference in the first quadrant only. By symmetry of the circle (how do we know this?) we can multiply our 1st quadrant result by (4) to get the contributions to the circumference, from the other three quadrants…

$$s = \int_0^R \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

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Since for our circle, $y = \sqrt{R^2 - x^2}$, the derivative is given by (show this yourself)…

$$\frac{dy}{dx} = \frac{-x}{\sqrt{R^2 - x^2}} \quad .$$

Substitute this derivative into the integrand… squaring it and adding (1). Using some algebra, simplify your result. You should get this…

$$s = R \bullet \int_0^R \frac{dx}{\sqrt{R^2 - x^2}}$$

Well, you can look up this integral, or you can solve it. Let's solve it ourselves…

First change variables to clean up the radicand.

Let $x = R \bullet u \quad \rightarrow \quad dx = R \bullet du$; for the limits, see that when x = 0, we have u = 0 ;

and when x = R, we have u = 1 . Our integral becomes…

$$s = R \bullet \int_0^1 \frac{R \bullet du}{R \bullet \sqrt{1-u^2}} \quad \rightarrow \quad s = R \bullet \int_0^1 \frac{du}{\sqrt{1-u^2}} \quad .$$

Now make another substitution… Let $u = Cos\phi \quad \rightarrow \quad du = -Sin\phi \bullet d\phi$; for the limits,

see that when u = 0, we have $\phi = Cos^{-1}(0) \quad \rightarrow \quad \phi = \frac{\pi}{2}$,

and when u = 1, we have $\phi = Cos^{-1}(1) \quad \rightarrow \quad \phi = 0$.

Show now, that the integral simplifies to…

$$s = R \bullet \int_{\frac{\pi}{2}}^{0} \frac{-Sin(\phi)d\phi}{Sin(\phi)}$$

$$s = -R \bullet [\phi]_{\frac{\pi}{2}}^{0} \quad \rightarrow \quad s = -R \bullet \left[0 - \frac{\pi}{2}\right] \quad \rightarrow \quad s = \frac{\pi R}{2}$$

But this last result is for the part of the circumference in the 1st quadrant. To get the full circumference, we must include the contributions from all four quadrants…

$$C = 4 \bullet s \quad \rightarrow \quad C = 4 \bullet \frac{\pi R}{2} \quad \rightarrow \quad C = 2\pi R \quad \text{or,} \quad C = \pi D .$$